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An analytical solution to the problem of thermoelastic bending of a multilayer beam with different temperature of longitudinal faces

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Abstract. An exact analytical solution of the quasistatic problem of thermoelasticity is presented for a section of a narrow multilayer beam with different temperatures of the longitudinal lower and upper faces and a heat flow of arbitrary height across the sections through the lateral faces. The solution was obtained for the entire package of layers by sequentially solving the heat equation for an inhomogeneous beam, taking into account the ideal thermal contact of the layers and the system of equations of the plane problem of the theory of elasticity under the assumption of a rigid connection of the layers. To take into account the inhomogeneity of the beam, a continuum approach is used, in which the multilayer material is considered continuous with variable physical and mechanical characteristics. The resulting relations take into account the orthotropy of the physico-mechanical properties of the materials of the layers and their compliance with the transverse shear and compression strains. An example of implementation of a solution for a five-layer beam with combined rigid and articulated fastening of the ends is given.

1. Introduction

Bars and beams are one of the most common elements of engineering structures. For composite beams, a multilayer structure with a pronounced difference in the mechanical characteristics of the layers is typical. With proper design, this provides a significant gain in the specific strength and stiffness of the composite beam in comparison with a homogeneous analogue. However, the combination of dissimilar materials in the structural element has certain disadvantages, one of which is the occurrence of temperature deformations and stresses. In contrast to homogeneous in multilayer boards even in a uniform temperature field, the occurrence of temperature stresses and displacements is inevitable. This can significantly reduce the reserve of the bearing capacity of the element for the perception of the useful external load. Therefore, the development of theories of bending of composite bars in the direction of taking into account temperature deformations is an important direction in the mechanics of structurally inhomogeneous elements.

Along with high-tech glass and carbon fiber, the reinforced concrete widely used in construction also belongs to composites. Despite the nonlinearity of the elastic characteristics of polymer matrices, reinforced plastics exhibit an almost linear relationship between stresses and strains up to failure [1]. As experimental studies show, for example [2, 3], when calculating reinforced concrete beams, it is important to take into account the nonlinearity of the mechanical characteristics of the concrete matrix. However, with certain reservations, the phase materials of such composites can be considered as linearly elastic materials.

To date, a significant number of scientific papers have been published on the problems of elastic deformation of composite beams with various methods of solution, types of loading and fastening, as well as features of the properties and structure of materials. If we consider the analytical direction of solution methods, then refined bending models are quite developed for multilayer beams, for example [4-7]. The main approaches to the construction of bending models of various orders can be found in the review [8].

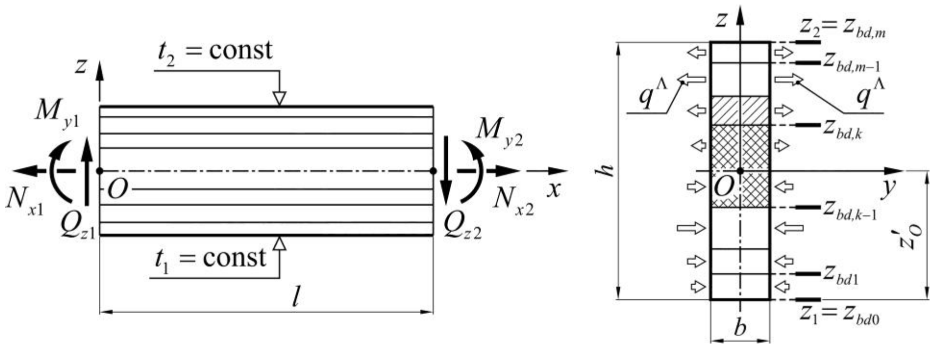
No less important for the analytical direction, is to obtain accurate solutions of the equations of elasticity theory for the problems of deformation of composite bars, which give a more accurate and complete picture of the distribution of stress-strain state (SSS) characteristics. Most of these solutions were obtained for a separate simple type of load: load at the end [9-13], uniform [9, 14-21], linear [22-25] or sinusoidal [26, 27] normal load on the longitudinal faces. In a number of works, more complex laws of changing the intensity of the load in the form of the sum of a power [28, 29] or trigonometric [30-32] series, as well as their combination [33], are considered.

Among the mentioned works, the majority is devoted to beams made of functionally gradient materials (FGM) and a relatively small number to multilayer beams [10, 11, 13, 20, 21, 25, 32]. The temperature component of stresses and displacements was considered only in [15, 35] devoted to FGM beams. If we do not take into account the classical theory of temperature bending of a bimetallic strip [36], which in some cases corresponds to the theory of elasticity, then for multilayer beams exact analytical solutions are practically absent.

In this work, an exact analytical solution is proposed for the quasistatic problem of thermoelasticity of a multilayer beam in an important case for practice of different temperatures of longitudinal surfaces and symmetric heat transfer through side faces. In this case, a complete system of equations of thermoelasticity is applied, including the heat equation.

2. Problem formulation

Let us consider a section of a multi-layer beam with a length l consisting of m heterogeneous layers of the same width b (figure 1). The layers are rigidly interconnected and have perfect thermal contact. The cross section of the beam is constant along the length of the section and has the shape of a narrow rectangle (figure 1b), the dimensions of which meet the condition: $b << h << l$.



a b

Figure 1. Scheme of a section of a bar and its cross section.

The longitudinal edges of the beam are free from external loads (figure 1a), and the forces in the extreme sections are reduced to the resultant $N_{x,\varsigma}, Q_{z,\varsigma}, M_{y,\varsigma}$ ($\varsigma = 1, 2$) and constitute a balanced system of forces.

We consider that during the manufacture of the beam, all its layers had the same temperature $T_0 = \text{const}$, and during operation on the surface of the lower ($z = z_1$) and upper ($z = z_2$) longitudinal faces, a uniform temperature $t_\varsigma = \text{const}$ is maintained. On the lateral longitudinal surfaces there is a symmetric heat exchange with the external environment. The heat flow density through these surfaces is constant along the length of the section and varies arbitrarily along the section height: $q^\Lambda = q^\Lambda(z)$.

The bar layers are homogeneous orthotropic or functionally gradient (FG) in thickness. The orthotropy planes of the mechanical and thermophysical characteristics are parallel to the coordinate planes. For an arbitrary k th layer, independent physical and mechanical characteristics are specified:

$$\left\| E_x^{[k]}, E_z^{[k]}, G_{xz}^{[k]}, \nu_{xz}^{[k]}, \vartheta_x^{[k]}, \vartheta_z^{[k]}, \lambda_{0x}^{[k]}, \lambda_{0z}^{[k]} \right\| = \left\| S_a^{[k]} \right\|, \quad (1)$$

where $E_x^{[k]}, E_z^{[k]}$ – longitudinal and transverse elastic moduli of k th layer; $G_{xz}^{[k]}$ – shear modulus; $\nu_{xz}^{[k]}$ – Poisson ratio; $\vartheta_x^{[k]}, \vartheta_z^{[k]}$ – linear thermal expansion coefficients; $\lambda_{0x}^{[k]}, \lambda_{0z}^{[k]}$ – thermal conductivity at initial temperature T_0.

Within a homogeneous layer, the physico-mechanical characteristics are constant $S_a^{[k]} = \text{const}$, and for the FG layer, the law of their change along the section height is specified $S_a^{[k]} = S_a^{[k]}(z)$.

For the entire section, an arbitrary characteristic, similar to [13, 20, 25, 31], can be represented as a piecewise continuous function $\mu_a^S(z)$:

$$\mu_a^S = \sum_{k=1}^{m} \left\{ S_a^{[k]} \left[H\left(z - z_{bd,k-1}\right) - H\left(z - z_{bd,k}\right) \right] \right\}, \tag{2}$$

where $H(z)$ – Heaviside function.

We consider that the axis Ox passes through the center of stiffness of each section of the bar, the coordinate z'_O (figure 1b) of which, relative to the lower edge of the section of the bar, is determined by the ratio

$$z'_O = \int_0^h \left(\mu_x'^E z \right) dz \Big/ \int_0^h \mu_x'^E dz, \tag{3}$$

where $\mu_x'^E$ – distribution function of the longitudinal elastic modulus for the case when the origin of the coordinate system is on the lower face of the section.

Let us determine the distribution of the temperature field and the field of stresses, deformations, and displacements associated with it within the considered section of the multilayer bar under the assumption that its thermal state is steady-state and upon transition to it the materials of all layers are deformed elastically.

3. The solution of heat conduction problem

When solving this problem, the priority is to obtain the distribution of the temperature field inside the beam, which gives the necessary initial data for determining its SSS.

In the case of a steady state of heat and in the absence of internal heat sources, the heat equation for a composite straight-line bar [34], after a corresponding replacement of the coordinate notation ($\eta \to x, \xi \to z$) and taking into account equation (2), will take the form

$$\mu_{0x}^{\lambda} \frac{\partial^2 T}{\partial x^2} + \frac{\partial}{\partial z}\left(\mu_{0z}^{\lambda} \frac{\partial T}{\partial z} \right) + \mu_{0y}^{\lambda} \frac{\partial^2 T}{\partial y^2} = 0, \tag{4}$$

where $T = T(x, y, z)$ – temperature field distribution function.

Integrating equation (4) over the section width and dividing the resulting expression by b, we have

$$\mu_{0x}^{\lambda} \frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial}{\partial z}\left(\mu_{0z}^{\lambda} \frac{\partial \overline{T}}{\partial z} \right) + \mu_{0y}^{\lambda} \frac{1}{b} \left(\frac{\partial T}{\partial y} \Big|_{y=y_1}^{y=y_2} \right) = 0, \tag{5}$$

where y_1, y_2 – the coordinates of the left and right longitudinal side faces of the bar; $\overline{T}$ – temperature averaged over the width of the section

$$\overline{T} = \frac{1}{b} \int_{y_1}^{y_2} T dy.$$

In the case of symmetric boundary conditions of the second kind on the lateral faces [34]:

$$\left(\mu_{0y}^{\lambda} \frac{\partial T}{\partial y} \right) \Big|_{y=y_\varsigma} = (-1)^{\varsigma+1} q^{\Lambda}, \ \varsigma = 1, 2, \tag{6}$$

where q^{Λ} – heat flow density through the side faces of the bar.

In view of equation (6), equation (5) takes the following form

$$\mu_{0x}^{\lambda}\frac{\partial^2\overline{T}}{\partial x^2}+\frac{\partial}{\partial z}\left(\mu_{0z}^{\lambda}\frac{\partial\overline{T}}{\partial z}\right)=\frac{2q^{\Lambda}}{b}. \tag{7}$$

Thus, we will seek the desired temperature distribution as a solution to equation (7), which corresponds to boundary conditions of the first kind on the longitudinal faces:

$$T|_{z=z_1}=t_1=\text{const}, \qquad T|_{z=z_2}=t_2=\text{const}. \tag{8}$$

The condition of symmetry of heat transfer conditions on the lateral longitudinal surfaces together with conditions equation (8) let us suggest that the solution to equation (7) will have the form $T=\overline{T}=\overline{T}(z)$. Then the partial differential equation (7) is transformed to an ordinary inhomogeneous differential equation

$$\frac{d}{dz}\left(\mu_{0z}^{\lambda}\frac{dT}{dz}\right)=\frac{2q^{\Lambda}}{b}. \tag{9}$$

By double integration equation (9) we obtain

$$T=\frac{2}{b}\int_{z_1}^{z}\left(\frac{1}{\mu_{0z}^{\lambda}}\int_{z_1}^{z}q^{\Lambda}dz\right)dz+C_0\int_{z_1}^{z}\frac{1}{\mu_{0z}^{\lambda}}dz+C_1, \tag{10}$$

where C_0, C_1 – unknown constants.

Substituting solution equation (10) into equation (8) and solving the obtained equations, we have

$$C_1=t_1, \qquad C_0=\frac{\Delta t-Q^{\Lambda}}{B_0^{\lambda}}, \tag{11}$$

where $B_0^{\lambda}, Q^{\Lambda}, \Delta t$ – constant:

$$B_0^{\lambda}=\int_{z_1}^{z_2}\frac{1}{\mu_{0z}^{\lambda}}dz, \qquad Q^{\Lambda}=\int_{z_1}^{z_2}\left(\frac{1}{\mu_{0z}^{\lambda}}\int_{z_1}^{z}q^{\Lambda}dz\right)dz, \qquad \Delta t=t_2-t_1. \tag{12}$$

In view of equation (11), solution equation (10) takes on following form

$$T=\frac{2}{b}\int_{z_1}^{z}\left(\frac{1}{\mu_{0z}^{\lambda}}\int_{z_1}^{z}q^{\Lambda}dz\right)dz+\frac{1}{B_0^{\lambda}}\left(\Delta t-\frac{2Q^{\Lambda}}{b}\right)\int_{z_1}^{z}\frac{1}{\mu_{0z}^{\lambda}}dz+t_1. \tag{13}$$

It should be noted that based on the general solution equation (10), one can also obtain particular solutions in the case of a given heat flow density (conditions of the second kind) or a given law of heat exchange with the environment (conditions of the third kind) on longitudinal faces [34].

Relation equation (13) gives the temperature field distribution inside the considered section of the multilayer beam and allows us to go on to determine the associated SSS. The obtained solution satisfies the heat equation and boundary conditions on the longitudinal surfaces and is accurate if the temperature distribution in the extreme sections of the section corresponds to equation (13).

4. The solution of the elasticity problem

The accepted initial conditions for determining the SSS of the considered section of the beam allow us to apply the equations of the plane problem of the linear theory of elasticity. The Navier equations and the Cauchy relations for the considered multilayer beam will be similar to the classical ones. However, instead of elastic constants, Hooke's law must contain functions equation (2), which allow one to take into account the inhomogeneity of the:

$$\varepsilon_x = \frac{1}{\mu_x^E}\left(\sigma_x - \mu_{xz}^{\nu}\sigma_z\right) + \mu_x^{\vartheta}\left(T - T_0\right), \qquad \varepsilon_z = \frac{\sigma_z}{\mu_z^E} - \frac{\mu_{xz}^{\nu}\sigma_x}{\mu_x^E} + \mu_z^{\vartheta}\left(T - T_0\right), \qquad \gamma_{xz} = \frac{1}{\mu_{xz}^G}\tau_{xz}, \quad (14)$$

where the dependence $\mu_x^E\mu_{zx}^{\nu} = \mu_z^E\mu_{xz}^{\nu}$ is taken into account for orthotropic material.

On the longitudinal edges of the beam, the solution must meet uniform static conditions

$$\sigma_z\big|_{z=z_\varsigma} = 0, \qquad \tau_{xz}\big|_{z=z_\varsigma} = 0,\ \varsigma = 1, 2. \quad (15)$$

In the leftmost section of the section, we require the fulfillment of integral conditions [13]:

$$\int_{z_1}^{z_2}\left(\sigma_x\big|_{x=0}\right)dz = \frac{N_{x1}}{b}, \qquad \int_{z_1}^{z_2}\left(z\sigma_x\big|_{x=0}\right)dz = -\frac{M_{y1}}{b}, \qquad \int_{z_1}^{z_2}\left(\tau_{xz}\big|_{x=0}\right)dz = -\frac{Q_{z1}}{b}. \quad (16)$$

Having solved the Navier equations with respect to normal stresses, taking into account equation (15), we obtain the following relations

$$\sigma_x = -\int_0^x \frac{\partial\tau_{xz}}{\partial z}dx + \sigma_x\big|_{x=0}, \qquad \sigma_z = -\int_{z_1}^{z}\frac{\partial\tau_{xz}}{\partial x}dz. \quad (17)$$

In the absence of load on the longitudinal faces of the bar, the transverse force: $Q_z = Q_{z1} = \text{const}$, which, with equation (16), allows us to make an assumption

$$\tau_{xz} = \tau_{xz}(z). \quad (18)$$

In view of equation (18), solutions of equation (17) take the form

$$\sigma_x = -\frac{d\tau_{xz}}{dz}x + \sigma_x\big|_{x=0}, \qquad \sigma_z = 0. \quad (19)$$

Substituting equation (19) into equation (14) we obtain the following relations for strains

$$\varepsilon_x = -\frac{1}{\mu_x^E}\left(\frac{d\tau_{xz}}{dz}x - \sigma_x\big|_{x=0}\right) + \mu_x^{\vartheta}\left(T - T_0\right),\ \varepsilon_z = \frac{\mu_{xz}^{\nu}}{\mu_x^E}\left(\frac{d\tau_{xz}}{dz}x - \sigma_x\big|_{x=0}\right) + \mu_z^{\vartheta}\left(T - T_0\right),\ \gamma_{xz} = \frac{\tau_{xz}}{\mu_{xz}^G}, \quad (20)$$

Having solved the Cauchy relations for linear strains with respect to longitudinal u and transverse w displacements, taking into account equation (20), we obtain

$$u = -\frac{1}{\mu_x^E}\left(\frac{d\tau_{xz}}{dz}\frac{x^2}{2} - \sigma_x\big|_{x=0}\,x\right) + \mu_x^{\vartheta}\left(T - T_0\right)x + u\big|_{x=0},$$
$$w = x\int_{z_1}^{z}\left(\frac{\mu_{xz}^{\nu}}{\mu_x^E}\frac{d\tau_{xz}}{dz}\right)dz - \int_{z_1}^{z}\left(\frac{\mu_{xz}^{\nu}}{\mu_x^E}\sigma_x\big|_{x=0}\right)dz + \int_{z_1}^{z}\left\{\mu_z^{\vartheta}\left(T - T_0\right)\right\}dz + w\big|_{z=z_1}. \quad (21)$$

Substituting equation (21) and the third dependence equation (20) into the Cauchy relation for angular strains, we obtain the following integral-differential expression

$$\frac{dw\big|_{z=z_1}}{dx} - \left[\frac{d}{dz}\left(\frac{1}{\mu_x^E}\frac{d\tau_{xz}}{dz}\right)\right]\frac{x^2}{2} + \left[\frac{d}{dz}\left(\frac{\sigma_x\big|_{x=0}}{\mu_x^E}\right) + \frac{d}{dz}\left\{\mu_x^{\vartheta}\left(T - T_0\right)\right\}\right]x +$$
$$+\left[\frac{du\big|_{x=0}}{dz} - \frac{\tau_{xz}}{\mu_{xz}^G} + \int_{z_1}^{z}\left(\frac{\mu_{xz}^{\nu}}{\mu_x^E}\frac{d\tau_{xz}}{dz}\right)dz\right] = 0. \quad (22)$$

Identity equation (22) will be valid for all points of the beam section only if the expressions in square brackets are equal to some constant:

$$\frac{d}{dz}\left(\frac{1}{\mu_x^E}\frac{d\tau_{xz}}{dz}\right)=K_0,\ \frac{d}{dz}\left(\frac{\sigma_x|_{x=0}}{\mu_x^E}\right)+\frac{d}{dz}\left\{\mu_x^9\left(T-T_0\right)\right\}=K_1,\ \frac{du|_{x=0}}{dz}-\frac{\tau_{xz}}{\mu_{xz}^G}+\int_{z_1}^{z}\left(\frac{\mu_{xz}^{\nu}}{\mu_x^E}\frac{d\tau_{xz}}{dz}\right)dz=K_2 \quad (23)$$

In view of equation (23), expression equation (22) is transformed into a linear differential equation:

$$\frac{dw|_{z=z_1}}{dx}=K_0\frac{x^2}{2}-K_1x-K_2 \quad (24)$$

Equations (23), (24) allow us to determine the desired function τ_{xz}, as well as unknown integration functions $\sigma_x|_{x=0}, u|_{x=0}, w|_{z=z_1}$.

When solving the first equation (23), taking into account the boundary conditions equation (15) and the third condition equation (16), we obtained

$$\tau_{xz}=-\frac{Q_{z1}}{bB_2}\int_{z_1}^{z}\left(\mu_x^E z\right)dz, \qquad K_0=-\frac{Q_{z1}}{bB_2}, \quad (25)$$

It is taken into accounts that in the adopted coordinate system [13]:

$$B_2=\int_{z_1}^{z_2}\int_{z_1}^{z}\left(\mu_x^E z\right)dzdz, \qquad \int_{z_1}^{z_2}\left(\mu_x^E z\right)dz=0 \quad (26)$$

Relation equation (25) is similar to the solution for shear stresses when bending a multilayer cantilever with a free end load. Thus, temperature strains do not directly affect the distribution of shear stresses and strains; however, they can affect their magnitude through a change in the shear force in statically indeterminate elements.

The general solution of the second equation (23)

$$\sigma_x|_{x=0}=-\mu_x^E\mu_x^9\left(T-T_0\right)+K_1\mu_x^E z+C_0\mu_x^E \quad (27)$$

Substituting equation (27) into conditions equation (16) and solving the obtained equations with respect to unknowns K_1 and C_0 taking into account equation (26), we have

$$C_0=\frac{N_{x1}+N^T}{bB_0}, \qquad K_1=\frac{M_{y1}-M^T}{bB_2}, \quad (28)$$

where N^T, M^T, B_0 – constant:

$$N^T=b\int_{z_1}^{z_2}\left[\mu_x^E\mu_x^9\left(T-T_0\right)\right]dz, \qquad M^T=b\int_{z_1}^{z_2}\left[\mu_x^E\mu_x^9 z\left(T-T_0\right)\right]dz, \qquad B_0=\int_{z_1}^{z_2}\mu_x^E dz \quad (29)$$

The quantities N^T and M^T have the dimension of force and moment, respectively. This allows us to consider them as the temperature components of the longitudinal force and bending moment inside the section of the beam.

Substituting equation (28) into equation (27) we obtain

$$\sigma_x|_{x=0}=\mu_x^E\left[\frac{N_{x1}+N^T}{bB_0}+\frac{M_{y1}-M^T}{bB_2}z-\mu_x^9\left(T-T_0\right)\right] \quad (30)$$

As we see in equation (30), the temperature component of normal stresses can be clearly distinguished, which at $T-T_0=0$ is equal to zero, which leads (30) to the relation obtained in [13].

The solution of the third equation (23) taking into account equation (25)

$$u|_{x=0}=-\frac{Q_{z1}}{bB_2}\int_{z_1}^{z}\left[\frac{1}{\mu_{xz}^G}\int_{z_1}^{z}\left(\mu_x^E z\right)dz-\int_{z_1}^{z}\left(\mu_{xz}^{\nu}z\right)dz\right]dz+K_2\left(z-z_1\right)+u|_{x=0,z=z_1} \quad (31)$$

Express the constant K_2 through the displacement of the upper fiber in the initial section. Accepting in equation (31) $z = z_2$ and solving relatively to K_2, we obtain

$$K_2 = \frac{Q_{z1}}{b}\frac{D_2}{hB_2} + \frac{1}{h}\left(u\,|_{x=0,z=z_2} - u\,|_{x=0,z=z_1}\right), \tag{32}$$

where D_2 – constant:

$$D_2 = \int_{z_1}^{z_2}\left[\frac{1}{\mu_{xz}^G}\int_{z_1}^{z}\left(\mu_x^E z\right)dz - \int_{z_1}^{z}\left(\mu_{xz}^\nu z\right)dz\right]dz. \tag{33}$$

In view of equation (32), relation (31) takes the following form

$$u\,|_{x=0} = -\frac{Q_{z1}}{bB_2}\left[\int_{z_1}^{z}\left(\frac{1}{\mu_{xz}^G}\int_{z_1}^{z}\left(\mu_x^E z\right)dz - \int_{z_1}^{z}\left(\mu_{xz}^\nu z\right)dz\right)dz - \frac{D_2}{h}\left(z - z_1\right)\right] + u\,|_{\substack{x=0,\\ z=z_2}}\frac{z - z_1}{h} + u\,|_{\substack{x=0,\\ z=z_1}}\frac{z_2 - z}{h}. \tag{34}$$

The solution of equation (24) after substituting (25), (28) and (32) is obtained in this form:

$$w\,|_{z=z_1} = -\frac{Q_{z1}}{bB_2}\left(\frac{x^3}{6} + \frac{D_2}{h}x\right) - \frac{M_{y1} - M^T}{bB_2}\frac{x^2}{2} - u\,|_{\substack{x=0,\\ z=z_2}}\frac{x}{h} + u\,|_{\substack{x=0,\\ z=z_1}}\frac{x}{h} + w\,|_{\substack{z=z_1,\\ x=0}}. \tag{35}$$

Solutions equations (25), (30), (34) and (35) together with (19), (20) and (21) allow us to write the final relations for the desired functions of stresses, deformations, and displacements.

Taking into account equations (19), (25) and (30) for stresses, we have

$$\sigma_x = \mu_x^E\left[\frac{Q_{z1}}{bB_2}zx + \frac{N_{x1} + N^T}{bB_0} + \frac{M_{y1} - M^T}{bB_2}z - \mu_x^\vartheta\left(T - T_0\right)\right], \qquad \sigma_z = 0, \qquad \tau_{xz} = -\frac{Q_{z1}}{bB_2}\int_{z_1}^{z}\left(\mu_x^E z\right)dz. \tag{36}$$

Strain equation (20) after substitution (25) and (30):

$$\begin{aligned}
\varepsilon_x &= \frac{Q_{z1}}{bB_2}zx + \frac{M_{y1} - M^T}{bB_2}z + \frac{N_{x1} + N^T}{bB_0},\\
\varepsilon_z &= -\mu_{xz}^\nu\left(\frac{Q_{z1}}{bB_2}zx + \frac{M_{y1} - M^T}{bB_2}z + \frac{N_{x1} + N^T}{bB_0}\right) + \left(\mu_{xz}^\nu\mu_x^\vartheta + \mu_z^\vartheta\right)\left(T - T_0\right),\\
\gamma_{xz} &= -\frac{Q_{z1}}{bB_2}\frac{1}{\mu_{xz}^G}\int_{z_1}^{z}\left(\mu_x^E z\right)dz.
\end{aligned} \tag{37}$$

Displacements equation (21) taking into account (34), (35):

$$\begin{aligned}
u = {}& \frac{Q_{z1}}{bB_2}\left[\frac{x^2 z}{2} - \int_{z_1}^{z}\left(\frac{1}{\mu_{xz}^G}\int_{z_1}^{z}\left(\mu_x^E z\right)dz - \int_{z_1}^{z}\left(\mu_{xz}^\nu z\right)dz\right)dz + \frac{D_2}{h}\left(z - z_1\right)\right] +\\
& + \frac{N_{x1} + N^T}{bB_0}x + \frac{M_{y1} - M^T}{bB_2}xz + u\,|_{\substack{x=0,\\ z=z_2}}\frac{z - z_1}{h} + u\,|_{\substack{x=0,\\ z=z_1}}\frac{z_2 - z}{h},\\
w = {}& -\frac{Q_{z1}}{bB_2}\left[\frac{x^3}{6} + x\left(\int_{z_1}^{z}\left(\mu_{xz}^\nu z\right)dz + \frac{D_2}{h}\right)\right] - \frac{M_{y1} - M^T}{bB_2}\left(\frac{x^2}{2} + \int_{z_1}^{z}\left(\mu_{xz}^\nu z\right)dz\right) -\\
& - \frac{N_{x1} + N^T}{bB_0}\int_{z_1}^{z}\mu_{xz}^\nu dz + \int_{z_1}^{z}\left[\left(\mu_{xz}^\nu\mu_x^\vartheta + \mu_z^\vartheta\right)\left(T - T_0\right)\right]dz + u\,|_{\substack{x=0,\\ z=z_1}}\frac{x}{h} - u\,|_{\substack{x=0,\\ z=z_2}}\frac{x}{h} + w\,|_{\substack{z=z_1,\\ x=0}}.
\end{aligned} \tag{38}$$

The conditions of absolutely rigid connection of the layers imply the equality of the total stress and displacement at each point on the common surface of arbitrary two layers, which requires the following static and kinematic conditions:

$$\tau_{xz}^{[k]}|_{z=z_{bd,k}} = \tau_{xz}^{[k+1]}|_{z=z_{bd,k}}, \qquad \sigma_z^{[k]}|_{z=z_{bd,k}} = \sigma_z^{[k+1]}|_{z=z_{bd,k}}, \; k=\overline{1,m-1}, \tag{39}$$

$$w^{[k]}|_{z=z_{bd,k}} = w^{[k+1]}|_{z=z_{bd,k}}, \qquad u^{[k]}|_{z=z_{bd,k}} = u^{[k+1]}|_{z=z_{bd,k}}. \tag{40}$$

In the solutions for τ_{xz}, u and w, the functions of the elastic characteristics μ_a^S, which have a discontinuity at the boundaries of the layers, are found only in integrand expressions. This ensures the continuity of the respective SSS components and the fulfillment of conditions equations (39) and (40).

Thus, relations equations (36), (37) and (38) satisfy all the equations of the plane problem of the theory of elasticity, boundary conditions on longitudinal surfaces, and conditions of absolutely rigid connection of layers. The resulting solution is accurate provided that in the extreme sections of the section the distribution of internal forces or external load corresponds to the obtained distribution of stresses.

The displacement solution contains three unknown displacements of the extreme points of the initial section of the beam section: $u|_{x=0,z=z_1}, u|_{x=0,z=z_2}, w|_{z=z_1,x=0}$ (initial kinematic parameters) and three forces in the initial section: N_{x1}, Q_{z1}, M_{y1} (initial static parameters). For a bar, which consists of several sections, these constants make it possible to coordinate the deformations of the considered section with its neighboring parts. In the case of a bar consisting of one section using these constants, you can simulate various types of fastening of the extreme sections.

In the general case, six unknowns $u|_{x=0,z=z_1}, u|_{x=0,z=z_2}, w|_{z=z_1,x=0}$ and N_{x1}, Q_{z1}, M_{y1} allow you to set independent movements of four arbitrary points of the extreme sections (two points in the left and right sections). In [21], a method for modeling various fastenings of the ends of a multilayer beam was proposed, which can be applied in the case under consideration.

Based on the principle of superposition, the obtained solution can be superimposed on the solutions [20, 25, 32], which allows one to obtain the SSS ratios for the case of multilayer beam bending by an arbitrary load on the longitudinal faces, taking into account the temperature conditions considered here.

5. Example of realization of the relations obtained

Let us determine the temperature field distribution and the resulting SSS for a five-layer beam with rigid fastening of the left end and pivotally fixed fastening of the right end (figure 2). The materials of the layers and their physical and mechanical characteristics:

- CFRP (P_1): $E_x^{[1]} = 142.80 GPa$, $E_z^{[1]} = 9.13 GPa$, $G_{xz}^{[1]} = 5.49 GPa$, $\nu_{xz}^{[1]} = 0.320$, $\vartheta_x^{[1]} = 0.0 K^{-1}$, $\vartheta_z^{[1]} = 27.7 \cdot 10^{-6} K^{-1}$, $\lambda_{0z}^{[1]} = 0.385\, W(m \cdot K)^{-1}$;
- GRP (P_2, P_4): $E_x^{[2,4]} = 36.8 GPa$, $E_z^{[2,4]} = 11.00 GPa$, $G_{xz}^{[2,4]} = 4.50 GPa$, $\nu_{xz}^{[2,4]} = 0.351$, $\vartheta_x^{[2,4]} = 7.8 \cdot 10^{-6} K^{-1}$, $\vartheta_z^{[2,4]} = 49.9 \cdot 10^{-6} K^{-1}$, $\lambda_{0z}^{[2,4]} = 0.460\, W(m \cdot K)^{-1}$;
- wood (P_3): $E_x^{[3]} = 12.80 GPa$, $E_z^{[3]} = 0.625 GPa$, $G_{xz}^{[3]} = 0.617 GPa$, $\nu_{xz}^{[3]} = 0.360$, $\vartheta_x^{[3]} = 5.4 \cdot 10^{-6} K^{-1}$, $\vartheta_z^{[3]} = 34.0 \cdot 10^{-6} K^{-1}$, $\lambda_{0z}^{[3]} = 0.107\, W(m \cdot K)^{-1}$;
- aluminum alloy (P_5): $E_x^{[5]} = E_z^{[5]} = 72.00 GPa$, $G_{xz}^{[5]} = 26.9 GPa$, $\nu_{xz}^{[5]} = 0.338$, $\vartheta_x^{[5]} = \vartheta_z^{[5]} = 22.9 \cdot 10^{-6} K^{-1}$, $\lambda_{0z}^{[5]} = 169.0\, W(m \cdot K)^{-1}$.

The initial temperature of the beam $T_0 = 293 K$. On the longitudinal surfaces of the beam constant temperature: $t_1 = 285K, t_2 = 273K$. Heat is exchanged through the lateral faces with the external environment, while the heat flow density is variable along the section height:

$$q^{\Lambda} = 12760.43z^2 + 66.57z - 11.94 \,. \tag{41}$$

The position of the center of stiffness of the section (figure 2b) is determined according to equation (3): $z'_O = 0.026078m$.

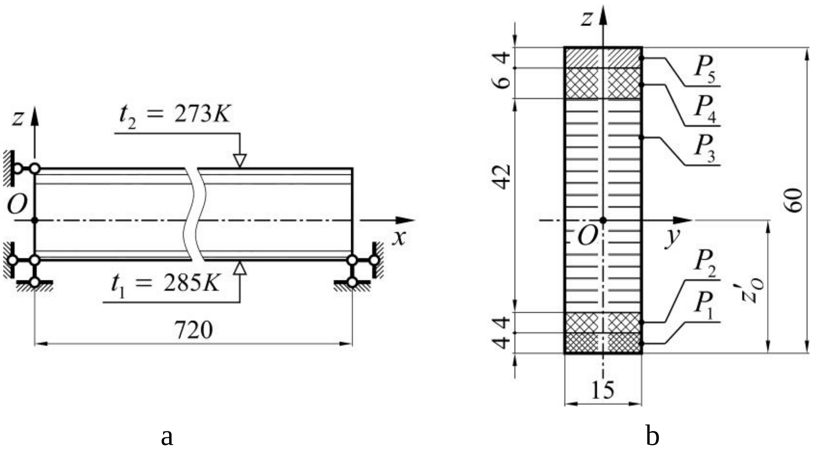


Figure 2. The scheme of kinematic restrictions and the cross-section of the beam (dimensions in mm).

The functions of physical and mechanical characteristics are formed according to equation (2):

$$\begin{aligned}\mu_a^S = S_a^{[1]}\left(H(z+0.02608)-H(z+0.02208)\right)+S_a^{[2]}\left(H(z+0.02208)-H(z+0.01808)\right)+\\ +S_a^{[3]}\left(H(z+0.01808)-H(z-0.02392)\right)+S_a^{[4]}\left(H(z-0.02392)-H(z-0.02992)\right)+\\ +S_a^{[5]}\left(H(z-0.02992)-H(z-0.03392)\right),\end{aligned} \tag{42}$$

Using equations (41), (42) according to equation (12), the constants necessary for determining the distribution of the temperature field are calculated:

$$B_0^{\lambda} = 0.42467\, m^2K/W, \qquad Q^{\Lambda} = -0.11762\, K\cdot m, \qquad \Delta t = -12\,K \,. \tag{43}$$

The desired temperature T distribution function was obtained according to equation (13) using equations (41), (42), (43) and given initial data. Graphs of the distribution of temperature T and heat flow density q_z over the section height along with the distributions $1/\mu_{0z}^{\lambda}$ and q^{Λ} are shown in figure 3.

The constants necessary for determining the SSS of the bar are determined according to equations (26), (29) and (33) using equation (42) and the obtained function T :

$$\begin{aligned}&B_0 = 1764.80\cdot 10^6\, N/m, \qquad B_2 = -9296.26\cdot 10^2\, N\cdot m, \qquad D_2 = -1244.11\cdot 10^{-6}\, m^3,\\ &N^T = -3130.29N, \qquad M^T = -77.3489\, N\cdot m.\end{aligned} \tag{44}$$

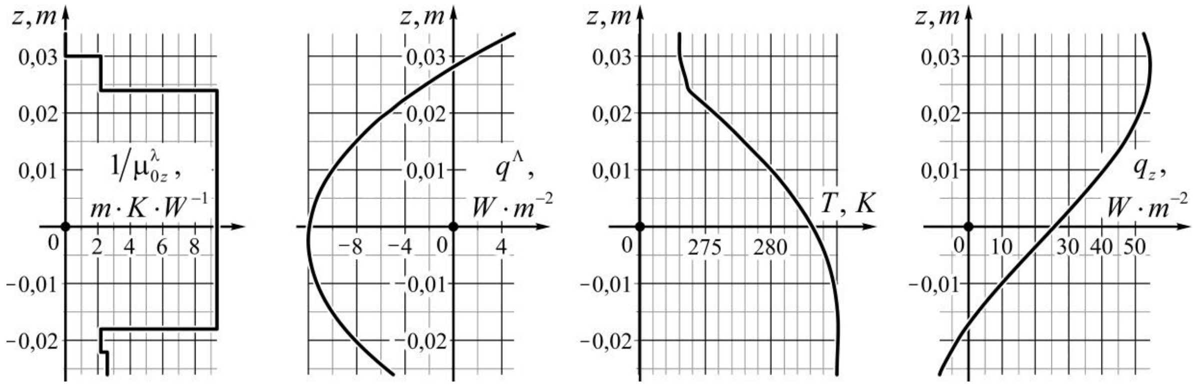


Figure 3. Initial data and results of solving the heat conduction problem

The problem under consideration is statically indeterminate; therefore, the kinematic ($u|_{x=0,z=z_1}, u|_{x=0,z=z_2}, w|_{z=z_1,x=0}$) and static ($N_{x1}, Q_{z1}, M_{y1}$) initial parameters are unknown. To determine them, we used the conditions of the static equilibrium of the beam and the kinematic conditions corresponding to the accepted types of fastenings.

Similarly to [21], rigid and simple fastening of the ends of the beam were modeled according to the diagram in figure 2a:

$$u|_{x=0,z=z_1} = 0, \qquad u|_{x=0,z=z_2} = 0, \qquad w|_{x=0,z=z_1} = 0, \qquad u|_{x=l,z=z_1} = 0, \qquad w|_{x=l,z=z_1} = 0 . \tag{45}$$

The missing equations are obtained from the conditions of static equilibrium

$$-N_{x1} + X_s = 0, \qquad Q_{z1} + Z_s = 0, \qquad -M_{y1} + Z_s l + X_s(-z_1) = 0 . \tag{46}$$

According to equation (45), the kinematic parameters are equal to zero, and the static ones are obtained by a joint solution of equations (45) and (46) with allowance for equation (38):

$$N_{x1} = 1307.59\,N, \qquad Q_{z1} = 207.31\,N, \qquad M_{y1} = -115.16\,N \cdot m . \tag{47}$$

Substituting the initial data, the corresponding functions equation (42), the constants equations (44), (45), (47) and the temperature distribution T in equations (36) and (38), we obtain the desired distribution functions of stresses and displacements.

Normal σ_x and shear τ_{xz} stress distributions for individual sections and fibers are shown in figure 4.

An abrupt change in the longitudinal elastic modulus at the boundaries of the layers (figure 4a) leads to discontinuities of the function σ_x. According to figure 4b, c, d the distribution of σ_x on the height of various sections is disproportionate, however, for each longitudinal fiber, in particular the outermost fibers (figure 4e), the normal stresses vary according to their own linear law. The shear stresses are constant along the length of the beam and vary according to a parabolic law along the thickness of each layer (figure 4f).

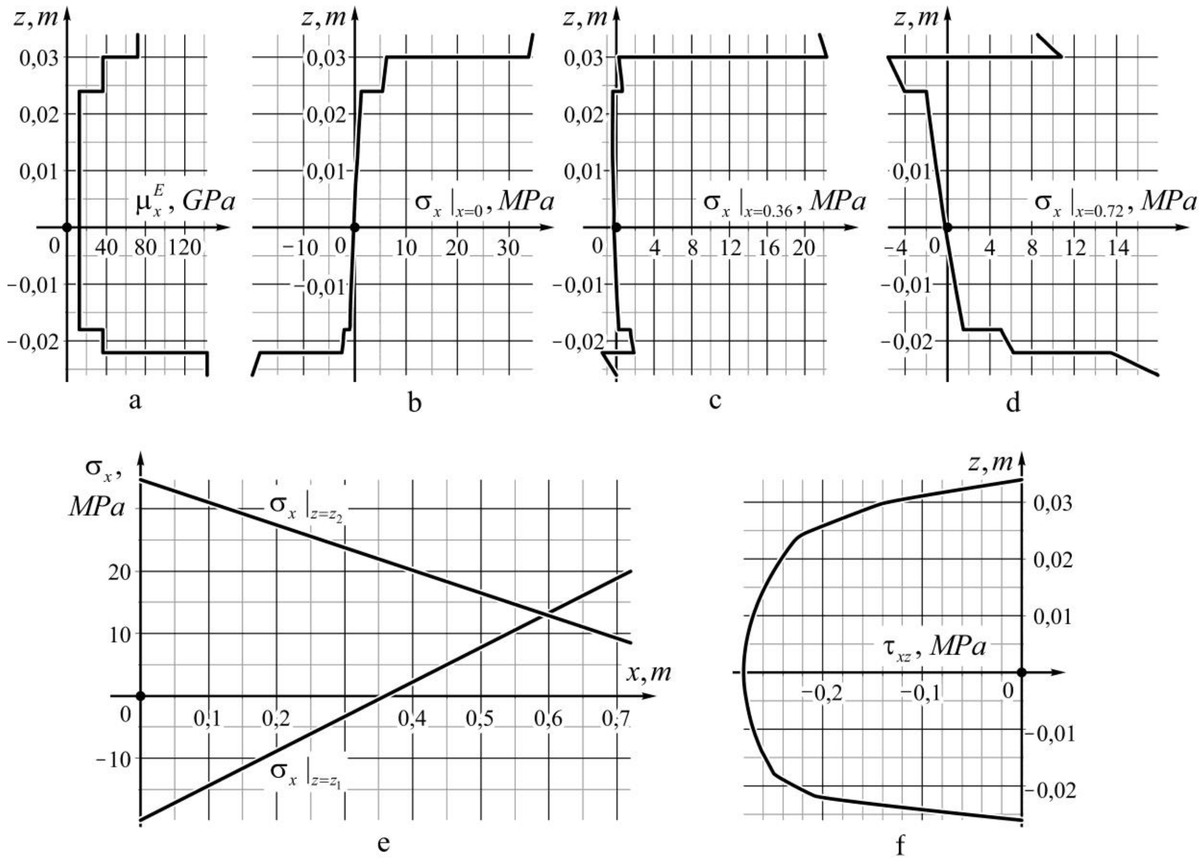


Figure 4. Graphs of normal and shear stresses.

Graphs for stresses σ_x show that even with a relatively small temperature change in the layers of the composite beam, significant tensile and compressive longitudinal forces can occur.

It is important to note the ambiguous effect of such stresses on the margin of strength of the beam. For example, under the action of a useful uniform load down, in the middle part of the beam, they will increase strength due to thermal tensile forces in the upper layer, but near the supports they will decrease on the contrary. Thus, for given temperature conditions of work of a multilayer beam by a certain alternation of layers, it is possible to increase its strength due to the effect of temperature redistribution of stresses between layers and sections.

The distributions of the longitudinal u and transverse w displacements for the extreme fibers and middle sections are shown in figure 5.

Graphs figure 5a, c shows the correspondence of the obtained solution to kinematic conditions equation (45).

Due to the presence of a transverse force in the graph of the distribution of longitudinal displacements u (figure 5b), one can observe a curvature of the cross sections, which is absent with other methods of securing the beam. The total curvature of the cross section is slightly enhanced by the displacement component of the displacement variable (figure 5d), which in this case is caused by the Poisson effect and thermal expansion of the layers.

The graphs of the transverse displacements w of the lower and upper fibers (figure 5c) show a slight decrease in the stiffness reserve of the beam for the case of a downward payload. This, obviously, can be avoided by changing the order of alternating layers.

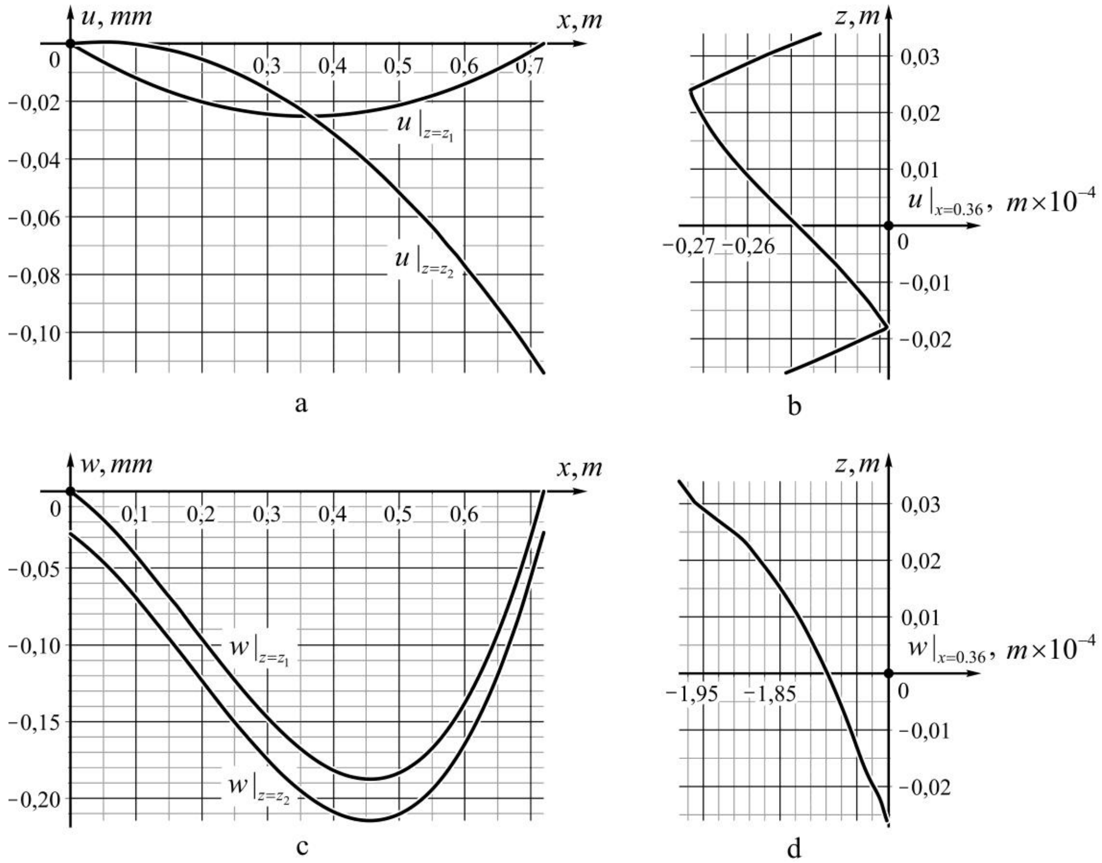


Figure 5. Graphs of longitudinal and transverse displacements.

6. Summary

Thus, an exact analytical solution to the problem of plane thermoelastic bending of a multilayer beam section has been constructed, which includes a solution to the heat conduction problem equation (13) and a solution to the theory of elasticity equations (36)-(38). The obtained solution allows us to determine the temperature field distribution inside the multilayer beam and the associated SSS with taking into account the different temperatures of its longitudinal surfaces and an arbitrary law of change in the heat flux density through the side faces. The solution is built for a beam with an arbitrary number of homogeneous or continuously inhomogeneous layers, it takes into account the orthotropy of the physical and mechanical characteristics and the flexibility of the transverse shear and compression of their materials.

The results of solving the test problem of thermoelastic bending of a five-layer bar with rigid and hinged fastening of the ends showed that a relatively small temperature change for such an element can cause significant stresses (up to 17% of the permissible) and significant deflections (14.8% of the permissible). Moreover, depending on the type of payload, the consideration of temperature stresses and displacements can lead to both a decrease and an increase in the strength and rigidity of the beam.

The obtained general solution can be used to refine the assessment of the strength and stiffness of multilayer beams taking into account temperature deformations, as well as to develop applied methods for calculating such elements.

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